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MOTION OF A NONRECTILINEAR FIBER IN A VISCOUS FLUID FLOW

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The plane problem of dynamic interaction of a laminar viscous fluid flow and an inextensible pliable fiber of finite length is solved using the perturbation method. Two types of rheological two-dimensional flows — pure shear and simple shear — are considered. Formulas are obtained for the evolution of the tensile force and the shape of the fiber. Results of asymptotic and numerical calculations are compared.

In process equipment (rollers and rubber mixers), the magnitude of strain of a filled system is finite because the time of mixing is limited [1–4]. Therefore, the initial fiber configuration has a significant influence on the fiber orientation at the end of production operations.

Fibrous fillers change markedly the rheological properties of fluids. For example, a viscous fluid filled with fibers exhibits abnormal properties: in the range of low velocities and shear stresses there is a sudden increase in viscosity. This effect is enhanced by increasing the length, concentration, and compliance of fibers [5].

The problem of the motion of a pliable fiber of finite length in a viscous fluid flow was formulated in [6]. For a fiber of a rectilinear shape, an analytical solution of the problem is obtained.

The goal of the present work was to study the effect of the initial curvature of the fiber on the rate of evolution of the fiber shape and tension.

1. Governing Equations. The following assumptions are adopted. The gravity and inertia are small. The fiber is isolated, i.e., there is no mechanical contact with the other fibers. The fiber does not introduce changes in the fluid velocity field. The flow is laminar and isothermal. The fiber axis remains a plane curve, and the condition $\max(d/l, kd) \ll 1$ is satisfied (d is the diameter of the fiber, $2l$ is its length, and k is the curvature). A frictional force proportional to the relative flow velocity acts on the fiber from the fluid.

Let us introduce the following dimensionless variables and parameters:

$$\tau = t|\dot\gamma g_1 + (1-g_1)\dot\gamma_-|, \qquad \{X, X_0, Y, Y_0, S\} = \{x, x_0, y, y_0, s\}l^{-1},$$
$$N = N_+/(A_\tau l^2|\dot\gamma g_1 + (1-g_1)\dot\gamma_-|), \qquad E = A_\tau/A_n. \tag{1.1}$$

Here t is time, $\dot\gamma$ is the pure shear strain rate, $\dot\gamma_-$ is the shear rate, g_1 is a parameter that characterizes the type of flow ($g_1 = 1$ corresponds to pure shear and $g_1 = 0$ corresponds to simple shear), $x(s)$, $y(s)$ is the equation of the fiber axis in parametric form (s is the coordinate along the fiber axis), $x_0(s)$ and $y_0(s)$ are functions that describe the initial configuration of the fiber axis, $A_n = 4\pi\mu/\ln(7.4/\mathrm{Re})$ [$\mathrm{Re} = \langle v\rangle\rho d/\mu$ is the Reynolds number, μ and ρ are the fluid viscosity and density, respectively, and $\langle v\rangle \approx |\dot\gamma g_1 + (1-g_1)\dot\gamma_-|l$ is the characteristic velocity], $A_\tau = 2.1\pi\mu\sqrt{\langle c\rangle}/\ln(0.952/\sqrt{\langle c\rangle})$, where $\langle c\rangle$ is the fiber volume fraction in the fluid, and N_+ is the fiber tension.

The equilibrium condition for the pliable fiber is written in vector form:

$$(N\boldsymbol{l})_s = -\boldsymbol{l}((\boldsymbol{v} - \boldsymbol{r}_\tau)\boldsymbol{l}) - \boldsymbol{n}((\boldsymbol{v} - \boldsymbol{r}_\tau)\boldsymbol{n})/E.$$

Here $\boldsymbol{r}$ is the radius-vector, $\boldsymbol{l} = \boldsymbol{r}_s$, $|\boldsymbol{l}| = 1$ is a unit vector directed along the tangent to the "elastic line" of the fiber, $\boldsymbol{r}_{ss} = -\varphi_s\boldsymbol{n}$, $\boldsymbol{n} = [\boldsymbol{lk}]$ is the unit vector of the principal normal, $\boldsymbol{k}$ is the unit vector parallel to the z axis, $\boldsymbol{v}$ is the fluid velocity, and $\boldsymbol{r}_\tau = d\boldsymbol{r}/d\tau$ is the velocity of the fiber. Solving this equation for $\boldsymbol{r}_\tau$, we obtain

$$\boldsymbol{r}_\tau = \boldsymbol{v} + N_s\boldsymbol{l} - EN\varphi_s\boldsymbol{n}.$$

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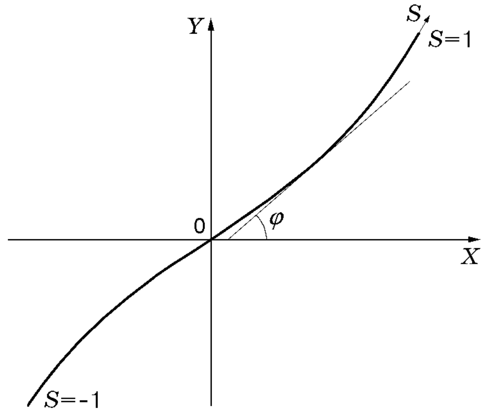

Fig. 1

Differentiating this equation with respect to s and taking into account that $\boldsymbol{n}_s = \varphi_s \boldsymbol{l}$ and $\boldsymbol{l}_\tau = \varphi_\tau \boldsymbol{n}$, we obtain

$$-\varphi_\tau \boldsymbol{n} = -(N_{ss} - EN\varphi_s^2)\boldsymbol{l} - (EN\varphi_{ss} + (1+E)N_s\varphi_s)\boldsymbol{n} + (\boldsymbol{l}\nabla)\boldsymbol{v}.$$

The equations of the problem considered are written in scalar form:

$$EN\,\frac{\partial^2\varphi}{\partial S^2} + (1+E)\,\frac{\partial N}{\partial S}\,\frac{\partial\varphi}{\partial S} - \frac{\partial\varphi}{\partial\tau} = g[g_1 \sin 2\varphi + (1-g_1)\sin^2\varphi],$$

$$\frac{\partial^2 N}{\partial S^2} - E\Big(\frac{\partial\varphi}{\partial S}\Big)^2 N = -g[g_1\cos 2\varphi + 0.5(1-g_1)\sin 2\varphi], \qquad (1.2)$$

$$\tau = 0\colon \quad \varphi = \varphi^*(S), \quad N = 0,$$

$$\tau > 0\colon \quad S = 0, \quad \frac{\partial\varphi}{\partial S} = \frac{\partial N}{\partial S} = 0, \quad S = 1, \quad \frac{\partial\varphi}{\partial S} = N = 0.$$

Here φ is the angle between the tangent to the fiber axis and the X axis, $\varphi^*(S)$ is the slope of the fiber axis at the initial moment, and g is a parameter that characterizes the flow direction ($g = \operatorname{sign}\dot\gamma$ for pure shear, and $g = \operatorname{sign}\dot\gamma_-$ for simple shear). It should be noted that in simple shear, the velocity field is characterized by the components $v_x = g|\dot\gamma_-|y$ and $v_y = 0$, and in pure shear, it is characterized by $v_x = g|\dot\gamma|x$ and $v_y = -g|\dot\gamma|y$.

The boundary conditions in (1.2) are written for the centrally symmetric initial configuration of the fiber. The middle of the fiber is at the coordinate origin, and the relations $X(S) = -X(-S)$ and $Y(S) = -Y(-S)$ hold. The point $S = 0$ is the point of inflection of the fiber axis at which $\partial\varphi/\partial S = 0$. In this case, for symmetric fluid-velocity fields, the middle of the fiber is always at the coordinate origin during fiber deformation (convective drift of the fiber). Therefore, it suffices to consider the motion of the right half of the fiber ($0 \leqslant S \leqslant 1$). The calculation scheme and the coordinate axes are shown in Fig. 1.

According to (1.1), the fluid viscosity determines tension but does not influence the evolution of the shape. Other things being equal, the fiber tension is proportional to the viscosity, the strain rate, and the squared fiber length, which agrees with experimental data. In the manufacture of rubber-fibrous compositions, the dispersion of fillers was enhanced by increasing the viscosity of the medium [1, 2], the strain rate [3], and the initial fiber length [4].

Thus, in studying the problem, we can confine ourselves to Eqs. (1.2). The functions X and Y are found by integration of the equations

$$\frac{\partial X}{\partial S} = \cos\varphi, \qquad \frac{\partial Y}{\partial S} = \sin\varphi, \qquad (1.3)$$

$$S = 0, \qquad X = 0, \qquad Y = 0.$$

2. Asymptotic Study. Let the initial configuration of the fiber be described by the function $\varphi^* = \varphi_+ + \varepsilon S$ (φ_+ and ε are constants).

Assuming $|\varepsilon| \ll 1$ in the initial conditions (1.2), we use the perturbation method [7] to analyze the problem. We seek a solution in the form of direct expansions in powers of the small parameter:

$$\varphi = \varphi_0(\tau) + \varepsilon\varphi_1(S,\tau) + \ldots, \qquad N = N_0(S,\tau) + \varepsilon N_1(S,\tau) + \ldots\ . \tag{2.1}$$

The first terms of the expansion (φ_0 and N_0) describe the evolution of a rectilinear fiber. We assume that $\varphi_0 = \varphi_0(\tau)$ because numerical analysis of the problem (1.2) showed that a rectilinear fiber retains its shape during evolution [6].

Substituting expansions (2.1) into the equations and boundary conditions (1.2) and equating terms of the same order of smallness in ε, we obtain the following problems:

— for order ε^0,

$$\frac{d\varphi_0}{d\tau} = -g[g_1 \sin 2\varphi_0 + (1-g_1)\sin^2\varphi_0],$$

$$\frac{\partial^2 N_0}{\partial S^2} = -g[g_1 \cos 2\varphi_0 + 0.5(1-g_1)\sin 2\varphi_0], \tag{2.2}$$

$$\tau = 0\colon\quad \varphi_0 = \varphi_+, \quad N_0 = 0,$$

$$\tau > 0\colon\quad S = 0, \quad \frac{\partial\varphi_0}{\partial S} = \frac{\partial N_0}{\partial S} = 0, \quad S = 1, \quad \frac{\partial\varphi_0}{\partial S} = N_0 = 0;$$

— for order ε^1,

$$EN_0\,\frac{\partial^2\varphi_1}{\partial S^2} + (1+E)\,\frac{\partial N_0}{\partial S}\,\frac{\partial\varphi_1}{\partial S} - \frac{\partial\varphi_1}{\partial\tau} = 2g\varphi_1[g_1\cos 2\varphi_0 + 0.5(1-g_1)\sin 2\varphi_0],$$

$$\frac{\partial^2 N_1}{\partial S^2} - EN_0\,\frac{\partial\varphi_1}{\partial S} = 2g\varphi_1[g_1 \sin 2\varphi_0 - 0.5(1-g_1)\cos 2\varphi_0], \tag{2.3}$$

$$\tau = 0\colon\quad \varphi_1 = S, \quad N_1 = 0,$$

$$\tau > 0\colon\quad S = 0, \quad \frac{\partial\varphi_1}{\partial S} = \frac{\partial N_1}{\partial S} = 0, \quad S = 1, \quad \frac{\partial\varphi_1}{\partial S} = N_1 = 0.$$

The solution of the problem (2.2) has the form

$$\varphi_0 = \arctan\,[\tan\,\varphi_+ \exp\,(-2g\tau)], \qquad N_0 = 0.5g(1-S^2)\cos 2\varphi_0 \tag{2.4}$$

in the case of pure shear ($g_1 = 1$) or

$$\varphi_0 = \arctan\,[\tan\,\varphi_+/(1 + g\tau\tan\,\varphi_+)], \quad g\tan\,\varphi_+ > 0, \qquad N_0 = 0.25g(1-S^2)\sin 2\varphi_0 \tag{2.5}$$

in the case of simple shear ($g_1 = 0$).

The solution of the first equation in (2.3) is sought in the form of the product of the functions

$$\varphi_1 = f(\tau)\psi(S), \tag{2.6}$$

which is a solution of the system

$$0.5E(1-S^2)\,\frac{d^2\psi}{dS^2} - (1+E)S\,\frac{d\psi}{dS} - (2+C)\psi = 0; \tag{2.7}$$

$$\frac{1}{f}\,\frac{df}{d\tau} = Cg[g_1\cos 2\varphi_0 + 0.5(1-g_1)\sin 2\varphi_0] \tag{2.8}$$

(C is a constant).

The solution of Eq. (2.7) has the form

$$\psi = C_1 S + C_2 S\int\limits_0^S S^{-2}(1-S^2)^{-(1+E)/E}\,dS,$$

where C_1 and C_2 are constants. The last integral with an arbitrary value of E is not expressed in terms of elementary functions. To satisfy the initial condition (2.3), it is necessary to set $C_1 = 1$, $C_2 = 0$, and $C = -(E+3)$.

In the case of pure shear ($g_1 = 1$), using the expression $d\varphi_0/d\tau$ from the first equation in (2.2), we can write Eq. (2.8) in the form

$$df/f = (E+3)\cot\,2\varphi_0\,d\varphi_0.$$

After integration, we obtain

$$f = (\sin 2\varphi_0 / \sin 2\varphi_+)^{(E+3)/2}. \tag{2.9}$$

Here we used the initial condition $\tau = 0$, $\varphi_0 = \varphi_+$, and $f = 1$.

Similarly, we obtain the solution of Eq. (2.8) for simple shear ($g_1 = 0$):

$$f = (\sin \varphi_0 / \sin \varphi_+)^{E+3}. \tag{2.10}$$

The solution of the second equation in (2.3) subject to (2.4)–(2.6), (2.9), and (2.10) is easy to obtain: double integration yields an expression for N_1.

Thus, for pure shear with accuracy up to terms $O(\varepsilon^2)$, we have the solution

$$\varphi = \varphi_0 + \varepsilon S (\sin 2\varphi_0 / \sin 2\varphi_+)^{(E+3)/2},$$

$$N = 0.5(1 - S^2) g \cos 2\varphi_0 + (\varepsilon g/24)(\sin 2\varphi_0 / \sin 2\varphi_+)^{(E+3)/2} \tag{2.11}$$

$$\times [E(6S^2 - S^4 - 5) \cos 2\varphi_0 - 8(1 - S^3) \sin 2\varphi_0],$$

where the function $\varphi_0(\tau)$ is defined in (2.4).

For simple shear, the relevant expressions have the form

$$\varphi = \varphi_0 + \varepsilon S (\sin \varphi_0 / \sin \varphi_+)^{E+3},$$

$$N = 0.25 g (1 - S^2) \sin 2\varphi_0 + (\varepsilon g/48)(\sin \varphi_0 / \sin \varphi_+)^{E+3} \tag{2.12}$$

$$\times [E(6S^2 - S^4 - 5) \sin 2\varphi_0 + 8(1 - S^3) \cos 2\varphi_0],$$

where the function $\varphi_0(\tau)$ is defined in (2.5).

Expansions (2.11) and (2.12) do not contain secular terms, which confirms the validity of expressions (2.1) and the analysis method used.

Integrating Eqs. (1.3) subject to (2.11) and (2.12), we obtain expressions for the functions X and Y:

$$X = [\sin(\varphi_0 + \varepsilon\alpha S) - \sin \varphi_0]/(\varepsilon\alpha), \qquad Y = [\cos \varphi_0 - \cos(\varphi_0 + \varepsilon\alpha S)]/(\varepsilon\alpha), \tag{2.13}$$

where $\alpha = g_1 (\sin 2\varphi_0 / \sin 2\varphi_+)^{(E+3)/2} + (1 - g_1)(\sin \varphi_0 / \sin \varphi_+)^{E+3}$; the function $\varphi_0(\tau)$ is defined in (2.4) and (2.5).

We consider a perfectly pliable fiber which is capable of transmitting only tensile forces and, therefore, the solution obtained is valid only for regions of stable motion of the fiber (see [6]).

During the evolution, the second terms in (2.11) and (2.12) decrease rapidly enough, and the fiber takes a rectilinear shape with a parabolic lengthwise distribution of tension. In this case, the parameter E, which determines the friction force, influences the rate of decrease of terms of order of ε but does not influence the evolution of the rectilinear fiber. Thus, the asymptotic analysis confirms the hypothesis on two periods of evolution of the fiber proposed in [6].

It should be noted that the parameter E depends on the volume fraction of the fibrous filler, and, therefore, its change in a production process influences the degree of fiber orientation.

Under pure shear, the originally rectilinear fiber, retaining its shape, performs a rotation around the point $X = Y = 0$ in the flow direction. A curve of tension versus the coordinate S is a parabola with vertex at the point $S = 0$. For $\tau \to \infty$ and $g = 1$, the axis of the fiber coincides with the X axis, and for $g = -1$, it coincides with the Y axis. In this case, the tension is maximal and is described by the dependence $N = 0.5(1 - S^2)$. The "effective longitudinal viscosity" of the system filled with fibers oriented in the tension direction is maximal. For fiber orientation $\varphi_0 = \pi/4$, the fiber tension is zero, and the effective viscosity of the system is close to the fluid viscosity.

In the case of simple shear, the nature of the tension distribution $N(S)$ changes significantly. For fiber orientation $\varphi_0 = g\pi/4$, the fiber tension is maximal and is described by the function $N = 0.25(1 - S^2)$. The shear viscosity of the filled system is maximal. In the case of equilibrium ($\tau \to \infty$), the fiber axis coincides with the X axis, and the tension is zeroth ($N = 0$). The shear viscosity of the filled system is close to the viscosity of the fluid (to a first approximation, the dependence of the viscosity on the fiber volume fraction is described by the Einstein formula).

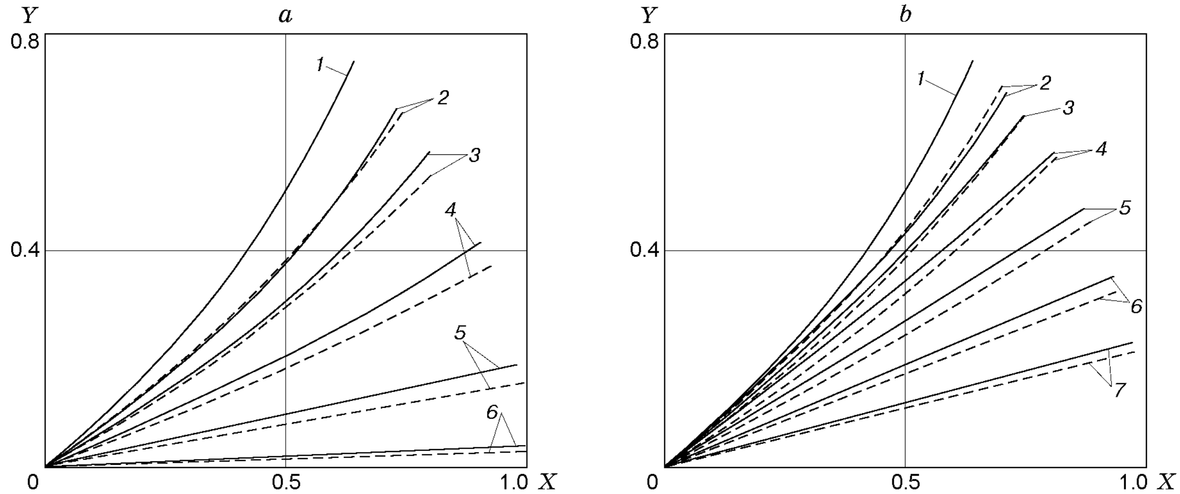

Fig. 2

3. Numerical Analysis. System (1.2), (1.3) was solved numerically using the Crank–Nicholson implicit finite-difference scheme. The functions N and φ in the upper time layer were obtained by the method of three-point marching and were refined by iterations. Here from Eqs. (1.3), we obtained the functions X and Y.

The scheme remained stable even in the presence of compressive forces in the fiber. In this case, in the compression region, the fiber assumed a saw-tooth shape with period approximately equal to two steps in S.

As a test, we used the exact solution of the problem for a rectilinear fiber (2.4), (2.5). The analysis was performed for the system "polycproamide fibers–rubber matrix" ($d = 30\ \mu$m, $2l = 10^{-2}$ m, $|\dot{\gamma}| = |\dot{\gamma}_-| = 18$ sec^{-1}, $\mu = 10^5$ Pa$\cdot$sec, $\rho = 1200$ kg/m^3, $\langle c\rangle = 0.05$, $\langle v\rangle = |\dot{\gamma}|l$, $\mathrm{Re} = 3.24 \cdot 10^{-8}$, and $E = 1.56$). The parameters of the initial configuration of the fiber were $\varphi_0 = 0.6$ and $\varepsilon = 0.5$. The step in the coordinate S was 0.025, and the step in time was 0.001.

Results of the numerical solution in dimensionless form are shown by solid lines in Fig. 2 (results of the asymptotic solution are shown by dashed curves). Figure 2a corresponds to pure shear, and Fig. 2b corresponds to simple shear. Curves 1–7 correspond to values $\tau = 0$, 0.1, 0.2, 0.4, 0.8, 1.6, and 3.2. It is evident that in the case of pure shear strain, the rotation of the fiber in the flow direction is performed at higher rate. Therefore, the presence of tensile strain in the flow accelerates the orientation of the fibrous filler.

The asymptotic solution was obtained under the assumption $|\varepsilon| \ll 1$. A comparison of this solution with the numerical solution shows that even for $\varepsilon = 0.5$, the asymptotic solution (2.11)–(2.13) describes the evolution of the fiber fairly well (difference not larger than 12%).

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